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Liapunov's direct method for Birkhoffian systems: Applications to electrical networks

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Abstract

In this paper, the concepts and the direct theorems of stability in the sense of Liapunov, within the framework of Birkhoffian dynamical systems on manifolds, are considered. The Liapunov-type functions are constructed for linear and nonlinear *LC* and *RLC* electrical networks, to prove stability under certain conditions.

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1. Introduction

During the last few years, a far reaching generalization of the Hamiltonian framework has been developed in a series of papers. This generalization, which is based on the geometric notion of generalized Dirac structure (see Courant [5] and Dorfman [6]), gives rise to implicit Hamiltonian systems (see, for example, the papers by Maschke and van der Schaft [12,14]). Applications to nonholonomic systems and electrical circuits (see Bloch and Crouch [2], Maschke and van der Schaft [12]) illustrate this theory. Recently, the notion of the implicit Lagrangian system has been developed by Yoshimura and Marsden [16]. Nonholonomic mechanical systems and degenerate Lagrangian systems such as *LC* circuits can be systematically formulated in the implicit Lagrangian context in which Dirac structures are also used.

An alternative approach to the study of dynamical systems is using the Birkhoffian formalism. This is a global formalism of implicit systems of second-order ordinary differential equations on a manifold. It applies to a wide class of systems, among them, nonholonomic systems, degenerate systems as well as dissipative systems. Kobayashi and Oliva developed in [9] the framework of Birkhoffian dynamical systems on manifolds, following Birkhoff's ideas presented locally in [1]. The space of configurations is a smooth m-dimensional differentiable connected manifold and the covariant character of the Birkhoff generalized forces is obtained by defining the notion of elementary work, called Birkhoffian, a special Pfaffian form defined on the 2-jet manifold. The dynamical system associated to this Pfaffian form is a subset of the 2-jet manifold which defines an implicit second-order ordinary differential system.

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The notion of Birkhoffian allows the introduction of the intrinsic concepts of reciprocity, regularity, affine structure in the accelerations, conservativeness [9], dissipativeness [7].

The Birkhoffian formalism in the context of electrical circuits was discussed by Ionescu and Scheurle [8] for the case of *LC* circuits, and Ionescu [7] for the case of *RLC* circuits. An *LC*/*RLC* circuit, with no assumptions placed on its topology, will be described by a family of Birkhoffian systems, parameterized by a finite number of real constants which correspond to initial values of certain state variables of the circuit. It is shown that the Birkhoffian system associated with an *LC* circuit is conservative. Under certain assumptions on the voltage–current characteristic for resistors, it is shown that a Birkhoffian system associated with an *RLC* circuit is dissipative. For *LC*/*RLC* networks which contain a number of loops formed only from capacitors, the Birkhoffian associated is never regular. A procedure for reducing the original configuration space to a lower dimensional one, thereby regularizing the Birkhoffian, is presented as well.

For *RLC* electrical networks, Brayton and Moser [3] proved, under a special hypothesis, that there exists a mixed potential function which can be used to put the system of differential equations describing the dynamics of such a network, into a special form (see Section 4 in [3]). The hypothesis they made is that the currents through the inductors and the voltages across the capacitors determine all currents and voltages in the circuit via Kirchhoff's law. The mixed potential function is constructed explicitly only for the networks whose graph possesses a tree containing all the capacitor branches and none of the inductive branches, that is, the network does not contain any loops of capacitors or cutsets of inductors, each resistor tree branch corresponds to a current-controlled resistor, each resistor co-tree branch corresponds to a voltage-controlled resistor (see Section 13 in [3]). Making different assumptions on the type of admissible nonlinearities in the circuit, this mixed potential function is used in [3] to construct Liapunov-type functions to prove stability.

Smale [15] also develops the differential equations for nonlinear *RLC* electrical circuits and illustrates these equations through a series of examples. He builds on the work of Brayton and Moser [3] but he is able to treat more general equations. A large part of the paper illustrates these equations by means of examples and discusses stability properties of the examples.

This paper is organized as follows. At the beginning of Section 2 we present the basics of Birkhoffian systems, from the viewpoint of differential geometry using the formalism of jets. Then, we introduce the concepts and the direct theorems of stability in the sense of Liapunov, within the framework of Birkhoffian systems. In Section 3 we consider, in turn, the linear and the nonlinear *LC* networks, as well as the linear and the nonlinear *RLC* networks. For each of them we construct Liapunov-type functions to prove stability or asymptotic stability under certain conditions. Finally, we discuss an example in Section 4.

2. Liapunov's direct method for Birkhoffian systems

In order to present the ideas in a coordinate free fashion, we consider the formalism of 2-jets. Let M be a m-dimensional differentiable connected manifold. We consider the tangent bundles (TM, π_M, M) and (TTM, π_{TM}, TM).

The *2-jet bundle* $(J^2(M), \pi_J, TM)$ is defined by

$$J^2(M) := \{z \in TTM / T\pi_M(z) = \pi_{TM}(z)\} \tag{2.1}$$

where $(T\pi_M)_v : T_v TM \to T_{\pi_M(v)}M$ is the tangent map and

$$\pi_J := \pi_{TM}\,|_{J^2(M)} = T\pi_M\,|_{J^2(M)}\,. \tag{2.2}$$

A local system of coordinates $(q) = (q^j)_{j=1,\dots,m}$ on M induces natural local coordinates on $J^2(M)$, denoted by $(q, \dot{q}, \ddot{q}) = (q^j, \dot{q}^j, \ddot{q}^j)_{j=1,\dots,m}$ (see for example [9,13]).

A *Birkhoffian* corresponding to the configuration manifold M is a smooth 1-form ω on $J^2(M)$ such that, for any $x \in M$, we have

$$\iota_x^* \omega = 0 \tag{2.3}$$

where $\iota_x : \beta^{-1}(x) \to J^2(M)$ is the embedding of the submanifold $\beta^{-1}(x)$ into $J^2(M)$, $\beta = \pi_M \circ \pi_J$. From this definition it follows that, in the natural local coordinate system $(q,\ \dot{q},\ \ddot{q})$ of $J^2(M)$, a Birkhoffian ω is given by

$$\omega = \sum_{j=1}^{m} \mathfrak{Q}_j(q,\, \dot{q},\, \ddot{q}) dq^j \tag{2.4}$$

with certain functions $\mathfrak{Q}_j : J^2(M) \to \mathbf{R}$. The pair $(M,\, \omega)$ is said to be a *Birkhoffian system* (see [9]).

The *differential system associated with a Birkhoffian* ω is the set (maybe empty) $D(\omega)$, given by

$$D(\omega) := \{z \in J^2(M) \ \mid\ \omega(z) = 0\}. \tag{2.5}$$

The manifold M is the *space of configurations* of $D(\omega)$, and $D(\omega)$ is said to have m 'degrees of freedom'. The $\mathfrak{Q}_i$ are the 'generalized external forces' associated with the local coordinate system. In the natural local coordinate system, $D(\omega)$ is characterized by the following implicit system of second-order ODE's:

$$\mathfrak{Q}_j(q,\, \dot{q},\, \ddot{q}) = 0 \quad \text{for all } j = \overline{1, m}. \tag{2.6}$$

The Birkhoffian formalism is a global formalism for the dynamics of implicit systems of second-order differential equations on a manifold.

A cross section X of the affine bundle $(J^2(M),\, \pi_J,\, TM)$, that is, a smooth function $X : TM \to J^2(M)$ such that $\pi_J \circ X = \text{id}$, can be identified with a special vector field on TM, namely, the second-order vector field Y on TM, that is, a smooth function $Y : TM \to TTM$ such that $\pi_{TM} \circ Y = \text{id}$ and $T\pi_M \circ Y = \text{id}$. Using the canonical embedding $i : J^2(M) \to TTM$, we write $Y = i \circ X$.

In natural local coordinates, a second-order vector field can be represented as

$$Y = \sum_{j=1}^{m} \left[\dot{q}^j \frac{\partial}{\partial q^j} + \ddot{q}^j(q,\, \dot{q}) \frac{\partial}{\partial \dot{q}^j} \right]. \tag{2.7}$$

A *Birkhoffian vector field* associated with a Birkhoffian ω of M (see [9]) is a smooth second-order vector field on TM, $Y = i \circ X$, with $X : TM \to J^2(M)$, such that $\text{Im}\, X \subset D(\omega)$, that is,

$$X^*\omega = 0. \tag{2.8}$$

In the natural local coordinate system, a Birkhoffian vector field is given by the expression (2.7), such that $\mathfrak{Q}_j(q, \dot{q}, \ddot{q}(q, \dot{q})) = 0$.

A Birkhoffian ω is *regular* if and only if

$$\det \left[\frac{\partial \mathfrak{Q}_j}{\partial \ddot{q}^i}(q,\, \dot{q},\, \ddot{q}) \right]_{j,i=1,\ldots,m} \neq 0 \tag{2.9}$$

for all $(q,\, \dot{q},\, \ddot{q})$, and for each $(q,\, \dot{q})$, there exists $(q, \dot{q}, \ddot{q}) \in J^2(M)$ such that $\mathfrak{Q}_j(q,\, \dot{q},\, \ddot{q}) = 0,\ j = 1, \ldots, m$.

If ω is a regular Birkhoffian corresponding to the configuration manifold M, then, *the principle of determinism* is satisfied, that is, there exists an *unique* Birkhoffian vector field $Y = i \circ X$ associated with ω such that $\text{Im}\, X = D(\omega)$ (see [9]).

A Birkhoffian ω of M is called *conservative* (see [9]) if and only if there exists a smooth function $E_\omega : TM \to \mathbf{R}$ such that

$$(X^*\omega)Y = \text{d}E_\omega(Y) \tag{2.10}$$

for all second-order vector fields $Y = i \circ X$, which is equivalent, in the natural local coordinate system, to the identity

$$\sum_{j=1}^{m} \mathfrak{Q}_j(q\ \dot{q},\, \ddot{q})\dot{q}^j = \sum_{j=1}^{m} \left[\frac{\partial E_\omega}{\partial q^j} \dot{q}^j + \frac{\partial E_\omega}{\partial \dot{q}^j} \ddot{q}^j \right]. \tag{2.11}$$

If ω is conservative and Y is a Birkhoffian vector field, then (2.10) becomes

$$\text{d}E_\omega(Y) = 0. \tag{2.12}$$

This means that E_ω is constant along the trajectories of Y.

A Birkhoffian ω of the configuration space M is called *dissipative* (see [7]) if and only if there exists a smooth function $E_{0_\omega} : TM \to \mathbf{R}$ such that

$$(X^*\omega)Y = \mathrm{d}E_{0_\omega}(Y) + D(Y) \tag{2.13}$$

for all second-order vector fields $Y = i \circ X$ on TM, D being a dissipative 1-form on TM, that is, $D = \sum_{j=1}^m D_j(q, \dot{q})dq^j$ and

$$\sum_{j=1}^{m} D_j(q, \dot{q})\dot{q}^j > 0. \tag{2.14}$$

Eq. (2.13) is equivalent, in a local coordinate system, to the identity

$$\sum_{j=1}^{m} \mathfrak{Q}_j(q\ \dot{q},\ \ddot{q})\dot{q}^j = \sum_{j=1}^{m} \left[\frac{\partial E_{0_\omega}}{\partial q^j}\dot{q}^j + \frac{\partial E_{0_\omega}}{\partial \dot{q}^j}\ddot{q}^j + D_j(q, \dot{q})\dot{q}^j \right]. \tag{2.15}$$

In view of (2.14), we obtain from (2.13),

$$(X^*\omega)Y > \mathrm{d}E_{0_\omega}(Y) \tag{2.16}$$

for all second-order vector fields $Y = i \circ X$. That is equivalent, in local coordinates, to the dissipation inequality

$$\sum_{j=1}^{m} \mathfrak{Q}_j(q\ \dot{q},\ \ddot{q})\dot{q}^j > \sum_{j=1}^{m} \left[\frac{\partial E_{0\omega}}{\partial q^j}\dot{q}^j + \frac{\partial E_{0_\omega}}{\partial \dot{q}^j}\ddot{q}^j \right]. \tag{2.17}$$

If ω is a dissipative Birkhoffian and Y is the Birkhoffian vector field, then (2.16) becomes

$$\mathrm{d}E_{0_\omega}(Y) < 0. \tag{2.18}$$

This means that E_{0_ω} is nonincreasing along the trajectories of Y.

Let us introduce now the concepts of stability for a Birkhoffian system.

The ***equilibrium points*** of the system, that is, the points in which the system can remain permanently at rest, are to be found as the solutions of the system

$$\mathfrak{Q}_j(q, 0, 0) = 0, \quad j = 1, \ldots, m. \tag{2.19}$$

Let us denote an equilibrium point by $(q_e, 0) \in \Omega \subset TM$, and an initial state of the system by $(q_0, \dot{q}_0)$, with $q(0) = q_0, \dot{q}(0) = \dot{q}_0$.

For regular Birkhoffians, we can define the equilibrium points using the notion of Birkhoffian vector field, that is, a point $(q_e, 0)$ is an equilibrium point of the Birkhoffian vector field Y if and only if

$$Y(q_e, 0) = 0. \tag{2.20}$$

An equilibrium point $(q_e, 0)$ is said to be **stable** (or Liapunov stable) if for every open neighborhood Ω of $(q_e, 0)$, there is a neighborhood $\Omega_1 \subset \Omega$ such that a motion $(q(t), \dot{q}(t))$ starting at $(q_0, \dot{q}_0) \in \Omega_1$, remains in Ω. If in addition, Ω_1 can be chosen such that, for any $(q_0, \dot{q}_0) \in \Omega_1$, $(q(t), \dot{q}(t))$ converges to $(q_e, 0)$ as $t \to \infty$, then $(q_e, 0)$ is said to be **asymptotically stable**.

In the memoir [11], Liapunov presents geometric theorems, generally referred to as the direct method of Liapunov (see, for example, [10]), for deciding the stability or instability of an equilibrium point of a differential equation.

In what follows we consider ***Liapunov's direct method for Birkhoffian systems***. This is based on finding a function $V \in C^1(TM, \mathbf{R})$ such that

$$\begin{aligned} &\text{(i) } V(q_e, 0) = 0 \\ &\text{(ii) } V(q, \dot{q}) > 0 \text{ for } (q, \dot{q}) \neq (q_e, 0) \text{ in } \Omega \\ &\text{(iii) } \mathrm{d}V(Y) \leq 0 \text{ for all second-order vector fields } Y \text{ defined on } \Omega \end{aligned} \tag{2.21}$$

with Ω an open neighborhood of $(q_e, 0)$. The function V is called ***Liapunov function***.

One can prove the following theorems (completely analogous to the theorems proved in [10] for a Liapunov function defined on $U \subset M$):

Stability Theorem. If there exists in a neighborhood Ω of $(q_e, 0)$ a Liapunov function $V(q, \dot{q})$, then $(q_e, 0)$ is stable.

Asymptotic Stability Theorem. If there exists in a neighborhood Ω of $(q_e, 0)$ a Liapunov function $V(q, \dot{q})$ such that $\mathrm{d}V(Y) < 0$ for all second-order vector fields Y defined on Ω, then $(q_e, 0)$ is asymptotically stable.

From the condition (ii) in (2.21) we get that there exists $c_0 > 0$ such that the level curve

$$\{(q, \dot{q}) \in \Omega, V(q, \dot{q}) = c\} \tag{2.22}$$

is a closed curve for every constant $0 \leq c \leq c_0$. Sketching in the m-plane $(q, \dot{q})$ these level curves of the function V, we obtain surfaces like "ellipsoids" centered at the equilibrium point.

If $\mathrm{d}V = 0$, then the equilibrium point $(q_e, 0)$ is a **center** and the motion of the system is periodic.

If $\mathrm{d}V < 0$, then each trajectory keeps moving to lower c and hence penetrates smaller and smaller "ellipsoids" as $t \to \infty$. Thus, the equilibrium point is asymptotically stable. This exclude the existence of periodic motions of the system.

3. Stability of the equilibrium points of *LC* and *RLC* networks

A simple electrical circuit provides us with an *oriented connected* graph. The graph will be assumed to be *planar*. Let b be the total number of branches in the graph, n be one less than the number of nodes and m be the cardinality of a selection of loops that cover the whole graph. By Euler's polyhedron formula, $b = m + n$. We choose a reference node and a current direction in each l-branch of the graph, $l = 1, \ldots, b$. We also consider a covering of the graph with m loops, and a current direction in each j-loop, $j = 1, \ldots, m$. We assume that the associated graph has at least one loop, meaning that $m > 0$. An oriented connected graph can be described by matrices which contain only 0, ± 1; these are: the *incidence matrix* $B \in \mathfrak{M}_{bn}(\mathbf{R})$, $\mathrm{rank}(B) = n$, and the *loop matrix* $A \in \mathfrak{M}_{bm}(\mathbf{R})$, $\mathrm{rank}(A) = m$. For the fundamentals of electrical circuit theory, see, for example, [4].

Let us now consider an *RLC* electrical circuit consisting of r resistors, k inductors and p capacitors, such that to each branch of the associated graph there corresponds just one electrical device, that is, $b = r + k + p$. For *LC* electrical circuits $r = 0$. Using the matrices A and B, Kirchhoff's current law and Kirchhoff's voltage law can be expressed by the equations

$$B^{\mathrm{T}}\mathrm{I} = 0 \quad (KCL), \quad A^{\mathrm{T}}v = 0 \quad (KVL) \tag{3.1}$$

where $\mathrm{I} = (\mathrm{I}_{[\Gamma]}, \mathrm{I}_{(a)}, \mathrm{I}_\alpha) \in \mathbf{R}^r \times \mathbf{R}^k \times \mathbf{R}^p \simeq \mathbf{R}^b$ is the current vector and $v = (v_{[\Gamma]}, v_{(a)}, v_\alpha) \in \mathbf{R}^r \times \mathbf{R}^k \times \mathbf{R}^p \simeq \mathbf{R}^b$ is the voltage drop vector. Tellegen's theorem establishes a relation between the matrices A^{T} and B^{T}: *the kernel of the matrix* B^{T} *is orthogonal to the kernel of the matrix* A^{T} (see, for example, page 5 of [3]).

We consider the voltage–current laws for nonlinear devices given by

$$v_{[\Gamma]} = R_\Gamma(\mathrm{I}_{[\Gamma]}), \quad v_{(a)} = L_a(\mathrm{I}_{(a)})\frac{\mathrm{dI}_{(a)}}{\mathrm{d}t}, \quad v_\alpha = C_\alpha(\mathrm{Q}_\alpha), \tag{3.2}$$

$R_\Gamma, L_a, C_\alpha : \mathbf{R} \longrightarrow \mathbf{R} \setminus \{0\}$ being smooth functions, Q_α denote the charges of the capacitors, with $\mathrm{I}_\alpha = \frac{\mathrm{dQ}_\alpha}{\mathrm{d}t}$. If the capacitors and the inductors are linear then the relations above become, respectively,

$$v_{[\Gamma]} = \mathrm{R}_\Gamma \mathrm{I}_{[\Gamma]}, \qquad v_{(a)} = \mathrm{L}_a \frac{\mathrm{dI}_{(a)}}{\mathrm{d}t} \qquad v_\alpha = \frac{\mathrm{Q}_\alpha}{\mathrm{C}_\alpha}, \tag{3.3}$$

where $\mathrm{R}_\Gamma \neq 0$, $\mathrm{C}_\alpha \neq 0$ and $\mathrm{L}_a \neq 0$ are distinct constants.

Summing up, the equations governing the network are

$$B^{\mathrm{T}} \begin{pmatrix} \mathrm{I}_{[\Gamma]} \\ \mathrm{I}_{(a)} \\ \dfrac{\mathrm{dQ}_\alpha}{\mathrm{d}t} \end{pmatrix} = 0, \qquad A^{\mathrm{T}} \begin{pmatrix} R_\Gamma(\mathrm{I}_{[\Gamma]}) \\ L_a(\mathrm{I}_{(a)})\dfrac{\mathrm{dI}_{(a)}}{\mathrm{d}t} \\ C_\alpha(\mathrm{Q}_\alpha) \end{pmatrix} = 0. \tag{3.4}$$

Using the first set of equations (3.4), one defines (see [7,8]) a family of m-dimensional affine–linear configuration spaces $M_{\mathfrak{c}} \subset \mathbf{R}^b$, parameterized by a constant vector $\mathfrak{c}$ in $\mathbf{R}^n$ which corresponds to initial values of certain state variables of the circuit. Since the matrix B is constant, integrating the first set of equations (3.4), one gets $B^{\mathrm{T}}x = \mathfrak{c}$, with $\mathrm{I} = \dot{x}$, $\mathfrak{c}$ a constant vector in $\mathbf{R}^n$. Thus, one defines

$$M_{\mathfrak{c}} := \{x \in \mathbf{R}^b | B^{\mathrm{T}}x = \mathfrak{c}\}. \tag{3.5}$$

Its dimension is $m = b - n$, because $\mathrm{rank}(B) = n$. Local coordinates on $M_{\mathfrak{c}}$ are denoted by $q = (q^1, \dots, q^m)$. Solving the system in (3.5), one expresses any of the x-variables in terms of qs, namely, as

$$x = \mathcal{N}q + \mathcal{K} \tag{3.6}$$

where $\mathcal{N} = \begin{pmatrix} \mathcal{N}_j^{\Gamma} \\ \mathcal{N}_j^{a} \\ \mathcal{N}_j^{\alpha} \end{pmatrix}_{\substack{\Gamma=\overline{1,r}, a=\overline{r+1,r+k}, \alpha=\overline{r+k+1,b}, \\ j=\overline{1,m}}}$ is a matrix of constants and $\mathcal{K} = \begin{pmatrix} \mathcal{K}^{\Gamma} \\ \mathcal{K}^{a} \\ \mathcal{K}^{\alpha} \end{pmatrix}$, a constant vector in $\mathbf{R}^b$.

By Tellegen's theorem and a fundamental theorem of linear algebra, one obtains that $\mathrm{Ker}(A^{\mathrm{T}}) = \mathrm{Ker}(\mathcal{N}^{\mathrm{T}})$ (see [7, 8]).

A Birkhoffian $\omega_{\mathfrak{c}}$ on the configuration space $M_{\mathfrak{c}}$ arises from a linear combination of the second set of equations (3.4), by replacing the matrix A^{T} with the matrix of constants $\mathcal{N}^{\mathrm{T}}$.

(I) For a linear LC network ($r = 0$) we have the following expression for the Birkhoffian (see [8]):

$$\mathfrak{Q}_j(q, \dot{q}, \ddot{q}) = \sum_{a=1}^{k} \sum_{i=1}^{m} \mathrm{L}_a \mathcal{N}_j^a \mathcal{N}_i^a \ddot{q}^i + \sum_{\alpha=k+1}^{b} \sum_{i=1}^{m} \frac{\mathcal{N}_j^{\alpha} \mathcal{N}_i^{\alpha}}{\mathrm{C}_{\alpha-k}} q^i + (\mathrm{const})_j \tag{3.7}$$

with const $\in \mathbf{R}^m$ a constant vector.

A linear LC network is *conservative* (see [8]). The function $E_{\omega} : TM_{\mathfrak{c}} \to \mathbf{R}$ satisfying (2.11) has the following expression:

$$E_{\omega}(q, \dot{q}) = \frac{1}{2} \sum_{a=1}^{k} \sum_{j,i=1}^{m} \mathrm{L}_a \mathcal{N}_j^a \mathcal{N}_i^a \dot{q}^j \dot{q}^i + \frac{1}{2} \sum_{\alpha=k+1}^{b} \sum_{j,i=1}^{m} \frac{\mathcal{N}_j^{\alpha} \mathcal{N}_i^{\alpha}}{\mathrm{C}_{\alpha-k}} q^j q^i + \sum_{j=1}^{m} (\mathrm{const})_j q^j. \tag{3.8}$$

In what follows we assume that

$$\det \left[\sum_{a=1}^{k} \mathrm{L}_a \mathcal{N}_j^a \mathcal{N}_i^a \right]_{j,i=1,\dots,m} \neq 0, \qquad \det \left[\sum_{\alpha=k+1}^{b} \frac{\mathcal{N}_j^{\alpha} \mathcal{N}_i^{\alpha}}{\mathrm{C}_{\alpha-k}} \right]_{j,i=1,\dots,m} \neq 0, \tag{3.9}$$

that is, the network does not contain loops formed only by capacitors and respectively, loops formed only by inductors (see [8]). If the network contains capacitor loops and inductor loops, we will first reduce the configuration space to a lower dimensional configuration space. On the reduced configuration space the corresponding Birkhoffian is still conservative (see [8]) and the corresponding determinants (3.9) will be different from zero. The inductor loops can be considered as some conserved quantities of the network.

Theorem 1. *Let $(q_e, 0)$ be an equilibrium point of a linear LC network with the Birkhoffian components given by (3.7). Then q_e satisfies the system*

$$\sum_{\alpha=k+1}^{b} \sum_{i=1}^{m} \frac{\mathcal{N}_j^{\alpha} \mathcal{N}_i^{\alpha}}{\mathrm{C}_{\alpha-k}} q^i + (\mathrm{const})_j = 0, \quad j = 1, \dots, m. \tag{3.10}$$

For each const *which is related to the initial data for the considered network, we get* ***a unique equilibrium point****. If*

$$\mathrm{L}_a > 0, \quad \forall a = 1, \dots, k, \qquad \mathrm{C}_{\alpha} > 0, \quad \forall \alpha = 1, \dots, p \tag{3.11}$$

the equilibrium point is a ***stable center****, and the motion of the system is periodic.*

Indeed, the equilibrium points of a linear LC network are obtained as solutions of the system $\mathfrak{Q}_j(q, 0, 0) = 0$, $j = 1, \ldots, m$, where $\mathfrak{Q}_j(q, \dot{q}, \ddot{q})$ is given by (3.7). Thus, we get that q_e has to fulfill the system (3.10). Under the second condition in (3.9), this system has for each const $\in \mathbf{R}^m$ a unique solution.

The stability of this equilibrium point is obtained by the Stability Theorem presented in Section 2. We define a Liapunov function $V \in C^1(TM_{\mathbf{c}}, \mathbf{R})$ by

$$\begin{aligned} V(q, \dot{q}) &= E_\omega(q, \dot{q}) - E_\omega(q_e, 0) \\ &= \frac{1}{2}\sum_{a=1}^{k}\sum_{j,i=1}^{m} \mathrm{L}_a \mathcal{N}_j^a \mathcal{N}_i^a \dot{q}^j \dot{q}^i + \frac{1}{2}\sum_{\alpha=k+1}^{b}\sum_{j,i=1}^{m} \frac{\mathcal{N}_j^\alpha \mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-k}} (q^j - q_e^j)(q^i - q_e^i) \end{aligned} \tag{3.12}$$

where q_e satisfies the system (3.10). Indeed, this function satisfies the conditions (2.21). Taking into account (3.11), the matrices $\left(\sum_{a=1}^{k} \mathrm{L}_a \mathcal{N}_j^a \mathcal{N}_i^a\right)_{j,i}$ and $\left(\sum_{\alpha=k+1}^{b} \frac{\mathcal{N}_j^\alpha \mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-k}}\right)_{j,i}$ are positive definite. Thus, the condition (ii) in (2.21) is fulfilled. The first determinant in (3.9) being different from zero implies that the corresponding Birkhoffian is regular. Therefore, along the trajectories of the unique (principle of determinism) Birkhoffian vector field, the function E_ω defined in (3.8) satisfies (2.12). Thus, the function (3.12) satisfies the condition (iii) in (2.21). In this case, sketching in the m-plane $(q, \dot{q})$ the level curves of the function (3.12), we obtain ellipsoids centered at the equilibrium point. The equilibrium point is a center and the motion of the system is periodic. □

(II) For a nonlinear LC network we have the following expression for the Birkhoffian (see [8]):

$$\begin{aligned} \mathfrak{Q}_j(q, \dot{q}, \ddot{q}) &= \sum_{a=1}^{k} \mathcal{N}_j^a L_a \left(\sum_{l=1}^{m} \mathcal{N}_l^a \dot{q}^l\right) \sum_{i=1}^{m} \mathcal{N}_i^a \ddot{q}^i + \sum_{\alpha=k+1}^{b} \mathcal{N}_j^\alpha C_{\alpha-k}\left(\sum_{l=1}^{m} \mathcal{N}_l^\alpha q^l + \mathcal{K}^\alpha\right) \\ &= \sum_{i=1}^{m}\left(\sum_{a=1}^{k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_a(\dot{q})\right)\ddot{q}^i + \sum_{\alpha=k+1}^{b} \mathcal{N}_j^\alpha \widetilde{C}_{\alpha-k}(q). \end{aligned} \tag{3.13}$$

A nonlinear LC network is *conservative* (see [8]). In this case, the function $E_\omega : TM_{\mathbf{c}} \to \mathbf{R}$ is given by

$$E_\omega(q, \dot{q}) = \mathcal{E}(\dot{q}) + \mathfrak{E}(q) \tag{3.14}$$

with

$$\begin{aligned} \mathcal{E}(\dot{q}) &= \sum_{a=1}^{k}\sum_{l=1}^{m}\sum_{i_1<\cdots<i_l=1}^{m} (-1)^{l+1} \underbrace{\int \cdots \int}_{l} \left[\widetilde{L}_a^{(l-1)}(\dot{q}) \mathcal{N}_i^a \dot{q}^i + (l-1)\widetilde{L}_a^{(l-2)}(\dot{q})\right] \mathcal{N}_{i_1}^a \ldots \mathcal{N}_{i_l}^a \mathrm{d}\dot{q}^{i_1} \ldots \mathrm{d}\dot{q}^{i_l} \\ \mathfrak{E}(q) &= \sum_{\alpha=k+1}^{b}\sum_{l=1}^{m}\sum_{i_1<\cdots<i_l=1}^{m} (-1)^{l+1} \underbrace{\int \cdots \int}_{l} \widetilde{C}_{\alpha-k}^{(l-1)}(q) \mathcal{N}_{i_1}^\alpha \ldots \mathcal{N}_{i_l}^\alpha \mathrm{d}q^{i_1} \ldots \mathrm{d}q^{i_l} \end{aligned} \tag{3.15}$$

where we defined the derivatives $\widetilde{C}_{\alpha-k}^{(l)} := \frac{\mathrm{d}^l \widetilde{C}_\alpha(x)}{\mathrm{d}x^l}$, $\widetilde{L}_a^{(l)} := \frac{\mathrm{d}^l \widetilde{L}_a(x)}{\mathrm{d}x^l}$.

In what follows we assume that

$$\det\left[\sum_{a=1}^{k} \mathcal{N}_j^a \mathcal{N}_i^a \widetilde{L}_a(\dot{q})\right]_{j,i=1,\ldots,m} \neq 0, \tag{3.16}$$

that is, the network does not contain capacitor loops. In the case where the network contains capacitor loops, we first reduce the configuration space to a lower dimensional one. On the reduced configuration space the corresponding Birkhoffian is still conservative (see [8]) and the corresponding determinant above will be different from zero.

Theorem 2. *Let $(q_e, 0)$ be an equilibrium point of a nonlinear LC network with the Birkhoffian components given by (3.13). Then q_e satisfies the system*

$$\sum_{\alpha=k+1}^{b} \mathcal{N}_j^{\alpha} C_{\alpha-k}\left(\sum_{l=1}^{m} \mathcal{N}_l^{\alpha} q^l + \mathcal{K}^{\alpha}\right) = 0, \quad j = 1, \ldots, m. \tag{3.17}$$

A nonlinear LC network can have ***several equilibrium points****. If*

$$L_a(0) > 0, \quad \forall a = 1, \ldots, k, \qquad C'_{\alpha}(q_e) > 0, \quad \forall \alpha = 1, \ldots, p \tag{3.18}$$

then the equilibrium points are ***locally stable centers****.*

Indeed, the equilibrium points of a nonlinear LC network are obtained as solutions of the system $\mathfrak{Q}_j(q, 0, 0) = 0$, $j = 1, \ldots, m$, where $\mathfrak{Q}_j(q, \dot{q}, \ddot{q})$ is given by (3.13). Thus, we see that q_e has to fulfill the system (3.17).

The local stability of the equilibrium points is obtained using the Stability Theorem presented in Section 2. We define a Liapunov function $V \in C^1(TM_{\mathfrak{c}}, \mathbf{R})$ by

$$V(q, \dot{q}) = E_{\omega}(q, \dot{q}) - E_{\omega}(q_e, 0) \tag{3.19}$$

with E_{ω} given by (3.14) and q_e satisfying the system (3.17).

Let us now evaluate the Hessian matrix of the function V in (3.19) at an equilibrium point $(q_e, 0)$. We get

$$\mathbf{H}_V(q_e, 0) = \begin{pmatrix} \dfrac{\partial^2 \mathcal{E}(\dot{q})}{\partial \dot{q}^i \partial \dot{q}^j}\Big|_{(q_e,0)} & 0 \\ 0 & \dfrac{\partial^2 \mathfrak{E}(q)}{\partial q^i \partial q^j}\Big|_{(q_e,0)} \end{pmatrix}. \tag{3.20}$$

For the Birkhoffian (3.13), the function E_{ω} in (3.14) satisfies the identity (2.11) (see [8]), that is,

$$\frac{\partial \mathcal{E}(\dot{q})}{\partial \dot{q}^i} = \sum_{a=1}^{k} \sum_{l=1}^{m} \widetilde{L}_a(\dot{q}) \mathcal{N}_i^a \mathcal{N}_l^a \dot{q}^l \tag{3.21}$$

$$\frac{\partial^2 \mathfrak{E}(q)}{\partial q^i} = \sum_{\alpha=k+1}^{b} \widetilde{C}_{\alpha-k}(q) \mathcal{N}_i^{\alpha}. \tag{3.22}$$

Therefore, we get

$$\frac{\partial^2 \mathcal{E}(\dot{q})}{\partial \dot{q}^i \partial \dot{q}^j} = \sum_{a=1}^{k} \left[\sum_{l=1}^{m} \widetilde{L}'_a(\dot{q}) \mathcal{N}_j^a \mathcal{N}_i^a \mathcal{N}_l^a \dot{q}^l + \widetilde{L}_a(\dot{q}) \mathcal{N}_i^a \mathcal{N}_j^a\right] \tag{3.23}$$

$$\frac{\partial^2 \mathfrak{E}(q)}{\partial q^i \partial q^j} = \sum_{\alpha=k+1}^{b} \widetilde{C}'_{\alpha-k}(q) \mathcal{N}_i^{\alpha} \mathcal{N}_j^{\alpha}. \tag{3.24}$$

From (3.23) and (3.24), the matrix (3.20) is written as

$$\mathbf{H}_V(q_e, 0) = \begin{pmatrix} \displaystyle\sum_{a=1}^{k} L_a(0) \mathcal{N}_i^a \mathcal{N}_j^a & 0 \\ 0 & \displaystyle\sum_{\alpha=k+1}^{b} \widetilde{C}'_{\alpha-k}(q_e) \mathcal{N}_i^{\alpha} \mathcal{N}_j^{\alpha} \end{pmatrix}. \tag{3.25}$$

In view of the conditions (3.18), the matrices $\left(\sum_{a=1}^{k} L_a(0) \mathcal{N}_i^a \mathcal{N}_j^a\right)_{i,j}$ and $\left(\sum_{\alpha=k+1}^{b} \widetilde{C}'_{\alpha-k}(q_e) \mathcal{N}_i^{\alpha} \mathcal{N}_j^{\alpha}\right)_{i,j}$ are positive definite. Therefore, the Hessian matrix (3.25) is positive definite. The centers of the level curves of the function (3.19) have the coordinates $(q_e, 0)$, where q_e satisfies the system (3.17). Thus, in a neighborhood of an equilibrium point, the condition (ii) in (2.21) is fulfilled by the function V in (3.19). The determinant (3.16) being different from zero implies that the corresponding Birkhoffian is regular. Therefore, along the trajectories of the unique (principle of determinism) Birkhoffian vector field, the function E_ω satisfies (2.12). Thus, the function (3.19) satisfies the condition (iii) in (2.21). By the Stability Theorem, the equilibrium points are locally stable centers. □

(III) For a linear *RLC* network we have the following expression for the Birkhoffian (see [7]):

$$\mathfrak{Q}_j(q,\dot q,\ddot q) = \sum_{a=r+1}^{r+k}\sum_{i=1}^{m} \mathrm{L}_{a-r}\mathcal{N}_j^a\mathcal{N}_i^a\ddot q^i + \sum_{\Gamma=1}^{r}\sum_{i=1}^{m}\mathrm{R}_\Gamma\mathcal{N}_j^\Gamma\mathcal{N}_i^\Gamma\dot q^i + \sum_{\alpha=r+k+1}^{b}\sum_{i=1}^{m}\frac{\mathcal{N}_j^\alpha\mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-r-k}}q^i + (\mathrm{const})_j, \tag{3.26}$$

with const $\in \mathbf{R}^m$ a constant vector.

A linear *RLC* network with

$$\mathrm{R}_\Gamma > 0, \quad \Gamma = 1,\dots,r, \tag{3.27}$$

is *dissipative* (see [7]). The function $E_{0_\omega} : TM_{\mathfrak{c}} \to \mathbf{R}$ and the dissipative 1-form satisfying (2.15) are given by

$$E_{0_\omega}(q,\dot q) = \frac{1}{2}\sum_{a=r+1}^{r+k}\sum_{j,i=1}^{m}\mathrm{L}_{a-r}\mathcal{N}_j^a\mathcal{N}_i^a\dot q^j\dot q^i + \frac{1}{2}\sum_{\alpha=r+k+1}^{b}\sum_{j,i=1}^{m}\frac{\mathcal{N}_j^\alpha\mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-r-k}}q^jq^i + \sum_{j=1}^{m}(\mathrm{const})_jq^j \tag{3.28}$$

$$D = \sum_{j,i=1}^{m}\sum_{\Gamma=1}^{r}\mathrm{R}_\Gamma\mathcal{N}_j^\Gamma\mathcal{N}_i^\Gamma\dot q^i dq^j. \tag{3.29}$$

In what follows we assume that

$$\det\left[\sum_{a=r+1}^{r+k}\mathrm{L}_{a-r}\mathcal{N}_j^a\mathcal{N}_i^a\right]_{j,i=1,\dots,m} \neq 0, \qquad \det\left[\sum_{\alpha=r+k+1}^{b}\frac{\mathcal{N}_j^\alpha\mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-r-k}}\right]_{j,i=1,\dots,m} \neq 0, \tag{3.30}$$

that is, the network does not contain capacitor loops and inductor loops, respectively. If the network contains capacitor loops and inductor loops, we will first reduce the configuration space to a lower dimensional configuration space. On the reduced configuration space the corresponding Birkhoffian is still dissipative (see [7]) and the corresponding determinants above will be different from zero.

Theorem 3. *Let* $(q_e, 0)$ *be an equilibrium point of a linear RLC network with the Birkhoffian given by* (3.26)*. Then* q_e *satisfies the system*

$$\sum_{\alpha=r+k+1}^{b}\sum_{i=1}^{m}\frac{\mathcal{N}_j^\alpha\mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-r-k}}q^i + (\mathrm{const})_j = 0, \quad j = 1,\dots,m. \tag{3.31}$$

For each const *which is related to the initial data for the considered network, we get* ***a unique equilibrium point****. If*

$$\mathrm{L}_a > 0, \quad \forall a = 1,\dots,k, \qquad \mathrm{C}_\alpha > 0, \quad \forall \alpha = 1,\dots,p, \tag{3.32}$$

the equilibrium point is ***asymptotically stable****.*

Indeed, the equilibrium points of a linear *RLC* network are obtained as solutions of the system $\mathfrak{Q}_j(q,0,0) = 0$, $j = 1,\dots,m$, where $\mathfrak{Q}_j(q,\dot q,\ddot q)$ is given by (3.26). Thus, we see that q_e has to fulfill the system (3.31). Under the second condition in (3.30), this system has for each const $\in \mathbf{R}^m$ a unique solution.

The asymptotic stability of this equilibrium point is obtained using the Asymptotic Stability Theorem presented in Section 2. We define a Liapunov function $V \in C^1(TM_{\mathfrak{c}}, \mathbf{R})$ by

$$V(q,\dot q) = E_{0_\omega}(q,\dot q) - E_{0_\omega}(q_e,0) = \frac{1}{2}\sum_{a=r+1}^{r+k}\sum_{j,i=1}^{m}\mathrm{L}_{a-r}\mathcal{N}_j^a\mathcal{N}_i^a\dot q^j\dot q^i + \frac{1}{2}\sum_{\alpha=r+k+1}^{b}\sum_{j,i=1}^{m}\frac{\mathcal{N}_j^\alpha\mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-r-k}}(q^j-q_e^j)(q^i-q_e^i) \tag{3.33}$$

where q_e satisfies the system (3.31). Indeed, this function satisfies the conditions (2.21). Taking into account (3.32), the matrices $\left(\sum_{a=r+1}^{r+k} \mathrm{L}_{a-r}\mathcal{N}_j^a\mathcal{N}_i^a\right)_{j,i}$ and $\left(\sum_{\alpha=r+k+1}^{b} \frac{\mathcal{N}_j^\alpha\mathcal{N}_i^\alpha}{\mathrm{C}_{\alpha-r-k}}\right)_{j,i}$ are positive definite. Thus, the condition (ii) in (2.21) is fulfilled. The first determinant in (3.30) being different from zero implies that the corresponding Birkhoffian is regular. Therefore, along the trajectories of the unique (principle of determinism) Birkhoffian vector field, the function E_{0_ω} satisfies (2.18). Thus, the function (3.33) also satisfies (2.18). In this case, sketching the level curves of the function (3.33) in the m-plane $(q, \dot{q})$, we obtain ellipsoids centered at the equilibrium point. From the Asymptotic Stability Theorem we conclude that the equilibrium point is asymptotically stable. This excludes the existence of periodic motions of the system. □

(IV) For a nonlinear *RLC* network we have the following expression for the Birkhoffian (see [7]):

$$\begin{aligned}\mathfrak{Q}_j(q, \dot{q}, \ddot{q}) &= \sum_{a=r+1}^{r+k} \mathcal{N}_j^a L_{a-r}\left(\sum_{l=1}^{m}\mathcal{N}_l^a\dot{q}^l\right)\left(\sum_{i=1}^{m}\mathcal{N}_i^a\ddot{q}^i\right)\\ &\quad+\sum_{\Gamma=1}^{r}\mathcal{N}_j^\Gamma R_\Gamma\left(\sum_{l=1}^{m}\mathcal{N}_l^\Gamma\dot{q}^l\right)+\sum_{\alpha=r+k+1}^{b}\mathcal{N}_j^\alpha C_{\alpha-r-k}\left(\sum_{l=1}^{m}\mathcal{N}_l^\alpha q^l+\mathcal{K}^\alpha\right)\\ &=\sum_{i=1}^{m}\sum_{a=r+1}^{r+k}\mathcal{N}_j^a\mathcal{N}_i^a\widetilde{L}_{a-r}(\dot{q})\ddot{q}^i+\sum_{\Gamma=1}^{r}\mathcal{N}_j^\Gamma\widetilde{R}_\Gamma(\dot{q})+\sum_{\alpha=r+k+1}^{b}\mathcal{N}_j^\alpha\widetilde{C}_{\alpha-r-k}(q).\end{aligned} \tag{3.34}$$

In order to obtain a *dissipative* Birkhoffian (see [7]), we assume that, for all $x \neq 0$,

$$xR_\Gamma(x) > 0, \quad \forall\Gamma = 1, \ldots, r \tag{3.35}$$

that is, for each nonlinear resistor, the graph of the function R_Γ lies in the first and third quadrants. The function $E_{0_\omega} : TM_{\mathbf{c}} \to \mathbf{R}$ and the dissipative 1-form satisfying (2.15) are given by

$$E_{0_\omega}(q, \dot{q}) = \mathcal{E}_0(\dot{q}) + \mathfrak{E}_0(q) \tag{3.36}$$

with

$$\begin{aligned}\mathcal{E}_0(\dot{q}) &= \sum_{a=r+1}^{r+k}\sum_{l=1}^{m}\sum_{i_1<\cdots<i_l=1}^{m}(-1)^{l+1}\underbrace{\int\int}_{l}^{\cdots}\left[\widetilde{L}_{a-r}^{(l-1)}(\dot{q})\mathcal{N}_i^a\dot{q}^i+(l-1)\widetilde{L}_{a-r}^{(l-2)}(\dot{q})\right]\mathcal{N}_{i_1}^a\cdots\mathcal{N}_{i_l}^a\mathrm{d}\dot{q}^{i_1}\cdots\mathrm{d}\dot{q}^{i_l}\\ \mathfrak{E}_0(q) &= \sum_{\alpha=r+k+1}^{b}\sum_{l=1}^{m}\sum_{i_1<\cdots<i_l=1}^{m}(-1)^{l+1}\underbrace{\int\int}_{l}^{\cdots}\widetilde{C}_{\alpha-r-k}^{(l-1)}(q)\mathcal{N}_{i_1}^\alpha\ldots\mathcal{N}_{i_l}^\alpha\mathrm{d}q^{i_1}\ldots\mathrm{d}q^{i_l}\end{aligned} \tag{3.37}$$

and

$$D = \sum_{j=1}^{m}\sum_{\Gamma=1}^{r}\mathcal{N}_j^\Gamma\widetilde{R}_\Gamma(\dot{q})\,\mathrm{d}q^j \tag{3.38}$$

In what follows we assume that

$$\det\left[\sum_{a=r+1}^{r+k}\widetilde{L}_{a-r}(\dot{q})\mathcal{N}_j^a\mathcal{N}_i^a\right]_{j,i=1,\ldots,m} \neq 0, \tag{3.39}$$

that is, the network does not contain capacitor loops. In the case where the network contains capacitor loops, first, we reduce the configuration space to a lower dimensional configuration space on which the corresponding Birkhoffian is still dissipative (see [7]) and on which the corresponding determinant above will be different from zero.

Theorem 4. *Let $(q_e, 0)$ be an equilibrium point of a nonlinear RLC network with the Birkhoffian given by (3.34). Then q_e satisfies the system*

$$\sum_{\Gamma=1}^{r} \mathcal{N}_j^{\Gamma} R_{\Gamma}(0) + \sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^{\alpha} C_{\alpha-r-k}\left(\sum_{l=1}^{m} \mathcal{N}_l^{\alpha} q^l + \mathcal{K}^{\alpha}\right) = 0, \quad j = 1, \dots, m. \tag{3.40}$$

A nonlinear RLC network can have ***several equilibrium points****.*

(1) *If*

$$R_{\Gamma}(0) = 0, \quad \forall \Gamma = 1, \dots, r \tag{3.41}$$

$$L_a(0) > 0, \quad \forall a = 1, \dots, k, \qquad C'_{\alpha}(q_e) > 0, \quad \forall \alpha = 1, \dots, p, \tag{3.42}$$

the equilibrium points are ***locally asymptotically stable****.*

(2) *If there exists $\Gamma = 1, \dots, r$ such that $R_{\Gamma}(0) \neq 0$, but, for all $x \neq 0$,*

$$x\,(R_{\Gamma}(x) - R_{\Gamma}(0)) > 0, \quad \forall \Gamma = 1, \dots, r \tag{3.43}$$

and the conditions (3.42) are fulfilled, then the equilibrium points are ***locally asymptotically stable****.*

Indeed, the equilibrium points of a nonlinear *RLC* network are obtained as solutions of the system $\mathfrak{Q}_j(q, 0, 0) = 0$, $j = 1, \dots, m$, where $\mathfrak{Q}_j(q, \dot{q}, \ddot{q})$ is given by (3.34). Thus, we see that q_e has to fulfill the system (3.40). The local asymptotic stability of the equilibrium points now follows from the Asymptotic Stability Theorem presented in Section 2.

First we assume condition (3.41) to be satisfied. Then the system (3.40) is written as

$$\sum_{\alpha=r+k+1}^{b} \mathcal{N}_j^{\alpha} C_{\alpha-r-k}\left(\sum_{l=1}^{m} \mathcal{N}_l^{\alpha} q^l + \mathcal{K}^{\alpha}\right) = 0, \quad j = 1, \dots, m. \tag{3.44}$$

In order to show (1), we define a Liapunov function $V \in C^1(TM_{\mathfrak{c}}, \mathbf{R})$ by

$$V(q, \dot{q}) = E_{0_\omega}(q, \dot{q}) - E_{0_\omega}(q_e, 0) \tag{3.45}$$

with E_{0_ω} given by (3.36) and q_e satisfying the system (3.44). In the neighborhood of any equilibrium point, this function satisfies the conditions (2.21). Taking into account the conditions (3.42), the Hessian matrix of the function V in (3.45), at the equilibrium point $(q_e, 0)$,

$$\mathbf{H}_V(q_e, 0) = \begin{pmatrix} \sum_{a=r+1}^{r+k} L_{a-r}(0)\mathcal{N}_i^a \mathcal{N}_j^a & 0 \\ 0 & \sum_{\alpha=r+k+1}^{b} \widetilde{C}'_{\alpha-r-k}(q_e)\mathcal{N}_i^{\alpha}\mathcal{N}_j^{\alpha} \end{pmatrix} \tag{3.46}$$

is positive definite. The centers of the level curves of the function (3.45) have the coordinates $(q_e, 0)$, where q_e satisfies the system (3.44). Thus, in a neighborhood of an equilibrium point the condition (ii) in (2.21) is fulfilled by the function V in (3.45). The determinant (3.39) being different from zero implies that the corresponding Birkhoffian is regular. Therefore, along the trajectories of the unique (principle of determinism) Birkhoffian vector field, the function E_{0_ω} satisfies (2.18). Thus, the function (3.45) also satisfies (2.18). By the Asymptotic Stability Theorem, the equilibrium points are locally asymptotically stable.

We assume now that there exists $\Gamma = 1, \dots, r$ such that $R_{\Gamma}(0) \neq 0$. In order to show (2), we consider instead of the function E_{0_ω} the following function $\mathrm{E}_{0_\omega} : TM_{\mathfrak{c}} \to \mathbf{R}$:

$$\mathrm{E}_{0_\omega}(q, \dot{q}) = \mathcal{E}_0(\dot{q}) + \mathfrak{E}_0(q) + \sum_{\Gamma=1}^{r} \mathcal{N}_j^{\Gamma} R_{\Gamma}(0) q^j \tag{3.47}$$

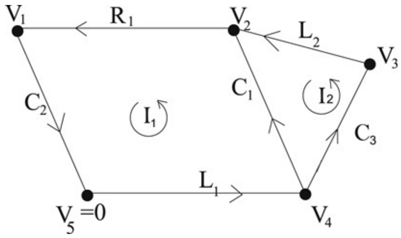

Fig. 1. An *RLC* circuit.

where $\mathcal{E}_0(\dot{q})$, $\mathfrak{E}_0(q)$ is given by (3.37), and instead of D the following dissipative 1-form:

$$\mathrm{D} = \sum_{j=1}^{m} \sum_{\Gamma=1}^{r} \mathcal{N}_j^{\Gamma} \left[\widetilde{R}_{\Gamma}(\dot{q}) - R_{\Gamma}(0)\right] \mathrm{d}q^j. \tag{3.48}$$

In view of assumption (3.43), the vertical 1-form (3.48) is indeed dissipative, that is,

$$\sum_{j=1}^{m} \sum_{\Gamma=1}^{r} \left(\mathcal{N}_j^{\Gamma} \dot{q}^j\right) \left[R_{\Gamma}\left(\sum_{l=1}^{m} \mathcal{N}_l^{\Gamma} \dot{q}^l\right) - R_{\Gamma}(0)\right] > 0. \tag{3.49}$$

One can easily check that for the function $\mathrm{E}_{0_\omega}(q, \dot{q})$ given by (3.47) and the dissipative 1-form (3.48), the Birkhoffian (3.34) is dissipative, that is, the identity

$$\sum_{j=1}^{m} \mathfrak{Q}_j(q\ \dot{q},\ \ddot{q})\dot{q}^j = \sum_{j=1}^{m} \left[\frac{\partial \mathrm{E}_{0_\omega}}{\partial q^j} \dot{q}^j + \frac{\partial \mathrm{E}_{0_\omega}}{\partial \dot{q}^j} \ddot{q}^j + \mathrm{D}_j(q, \dot{q})\dot{q}^j\right] \tag{3.50}$$

is fulfilled.

We define now a Liapunov function $\mathrm{V} \in C^1(TM_{\mathfrak{c}}, \mathbf{R})$ by

$$\mathrm{V}(q, \dot{q}) = \mathrm{E}_{0_\omega}(q, \dot{q}) - \mathrm{E}_{0_\omega}(q_e, 0). \tag{3.51}$$

If ω is a dissipative Birkhoffian and Y is the Birkhoffian vector field, then (2.18) becomes

$$\mathrm{d}\mathrm{E}_{0_\omega}(Y) < 0. \tag{3.52}$$

The function V in (3.51) satisfies (3.52) as well.

The centers of the level curves of the function (3.51) have the coordinates $(q_e, 0)$, where q_e satisfies the system (3.40). The Hessian matrix of the function V in (3.51) has at the equilibrium point the same expression (3.46). By the Asymptotic Stability Theorem, the equilibrium points are locally asymptotically stable. □

4. Example

We consider an electrical circuit with an associated oriented connected graph as in Fig. 1.

We have $r = 1$, $k = 2$, $p = 3$, $n = 4$, $m = 2$, $b = 6$. We choose the reference node to be V_5 and the current directions as indicated in Fig. 1. We cover the associated graph with the loops I_1, I_2. The branches in Fig. 1 are labelled as follows: the first branch is the resistive branch R_1, the second and the third branches are the inductive branches L_1, L_2 and the last three branches are the capacitor branches C_1, C_2, C_3. The incidence and loop matrices, $B \in \mathfrak{M}_{64}(\mathbf{R})$ and $A \in \mathfrak{M}_{62}(\mathbf{R})$, are written as

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

One has $\text{rank}(B) = 4$, $\text{rank}(A) = 2$. Kirchhoff's current law and Kirchhoff's voltage law can be expressed by the equations

$$B^{\mathrm{T}}\mathrm{I} = 0 \quad (KCL), \quad A^{\mathrm{T}}v = 0 \quad (KVL) \tag{4.2}$$

where $\mathrm{I} = (\mathrm{I}_{[\Gamma]}, \mathrm{I}_{(a)}, \mathrm{I}_\alpha) \in \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^3$ and $v = (v_{[\Gamma]}, v_{(a)}, v_\alpha) \in \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^3$ is the voltage drop vector.

We define the configuration space by

$$M_{\mathfrak{c}} := \{x \in \mathbf{R}^6 | B^{\mathrm{T}}x = \mathfrak{c}\} \tag{4.3}$$

with $\mathfrak{c}$ a constant vector in $\mathbf{R}^4$. $M_{\mathfrak{c}}$ is an affine–linear subspace in $\mathbf{R}^6$; its dimension is 2. The system in (4.3) is written as

$$\begin{aligned} &x^1 - x^5 = \mathfrak{c}_1 \\ &-x^1 + x^3 + x^4 = \mathfrak{c}_2 \\ &-x^3 + x^6 = \mathfrak{c}_3 \\ &x^2 - x^4 - x^6 = \mathfrak{c}_4. \end{aligned} \tag{4.4}$$

We denote local coordinates on $M_{\mathfrak{c}}$ by $q = (q^1, q^2)$. If we take, for example,

$$q^1 := x^5, \qquad q^2 := x^6 \tag{4.5}$$

we get

$$\begin{aligned} &x^1 = q^1 + \mathfrak{c}_1 \\ &x^2 = q^1 + \mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3 + \mathfrak{c}_4 \\ &x^3 = q^2 - \mathfrak{c}_3 \\ &x^4 = q^1 - q^2 + \mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3. \end{aligned} \tag{4.6}$$

Thus, the matrix of constants $\mathcal{N}$ in (3.6) is exactly the matrix A and the constant

$$\mathcal{K} = \begin{pmatrix} \mathfrak{c}_1 \\ \mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3 + \mathfrak{c}_4 \\ -\mathfrak{c}_3 \\ \mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3 \\ 0 \\ 0 \end{pmatrix}.$$

First we consider the case where all the electrical devices in the circuit are linear; they are described by the relations (3.3). In this case, in terms of the q-coordinates (4.5), the Birkhoffian $\omega_{\mathfrak{c}}$ on $M_{\mathfrak{c}}$ is written as

$$\begin{aligned} &\mathfrak{Q}_1(q, \dot{q}, \ddot{q}) = \mathrm{L}_1\ddot{q}^1 + \mathrm{R}_1\dot{q}^1 + \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2}\right) q^1 - \frac{1}{\mathrm{C}_1} q^2 + \frac{\mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3}{\mathrm{C}_1} \\ &\mathfrak{Q}_2(q, \dot{q}, \ddot{q}) = \mathrm{L}_2\ddot{q}^2 - \frac{1}{\mathrm{C}_1} q^1 + \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_3}\right) q^2 - \frac{\mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3}{\mathrm{C}_1}. \end{aligned} \tag{4.7}$$

Let us see now how the constants are related to the initial conditions that may be specified for the considered network.

The differential system associated with the Birkhoffian (4.7) is written

$$\mathfrak{Q}_1(q, \dot{q}, \ddot{q}) = 0, \qquad \mathfrak{Q}_2(q, \dot{q}, \ddot{q}) = 0. \tag{4.8}$$

For each capacitor we are able to specify the initial charge, that is, $\mathrm{Q}_1(0)$, $\mathrm{Q}_2(0)$, $\mathrm{Q}_3(0)$, and for each inductor the initial current, that is, $\mathrm{I}_{(1)}(0)$, $\mathrm{I}_{(2)}(0)$. Taking into account (4.5), the relation $\mathrm{I} = \dot{x}$ and the second and third relations in (4.6), we have the following initial conditions for the differential system (4.8):

$$\begin{aligned} &q^1(0) = \mathrm{Q}_2(0)\\ &q^2(0) = \mathrm{Q}_3(0)\\ &\dot{q}^1(0) = \mathrm{I}_{(1)}(0)\\ &\dot{q}^2(0) = \mathrm{I}_{(2)}(0). \end{aligned} \tag{4.9}$$

Besides, taking into account the (4.5) and the last relation in (4.6), we find

$$\mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3 = \mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0). \tag{4.10}$$

Thus, the Birkhoffian (4.7) becomes

$$\begin{aligned} \mathfrak{Q}_1(q,\dot{q},\ddot{q}) &= \mathrm{L}_1\ddot{q}^1 + \mathrm{R}_1\dot{q}^1 + \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2}\right)q^1 - \frac{1}{\mathrm{C}_1}q^2 + \frac{\mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)}{\mathrm{C}_1}\\ \mathfrak{Q}_2(q,\dot{q},\ddot{q}) &= \mathrm{L}_2\ddot{q}^2 - \frac{1}{\mathrm{C}_1}q^1 + \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_3}\right)q^2 - \frac{\mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)}{\mathrm{C}_1}. \end{aligned} \tag{4.11}$$

If the constant $\mathrm{R}_1 > 0$, the Birkhoffian (4.11) is **dissipative**. The function $E_{0_\omega} : TM_{\mathfrak{c}} \to \mathbf{R}$ and the dissipative 1-form satisfying (2.15) have the expressions

$$\begin{aligned} E_{0_\omega}(q,\dot{q}) \;=\; &\frac{1}{2}\mathrm{L}_1(\dot{q}^1)^2 + \frac{1}{2}\mathrm{L}_2(\dot{q}^2)^2 + \frac{1}{2\mathrm{C}_1}(q^1 - q^2)^2 + \frac{1}{2\mathrm{C}_2}(q^1)^2 + \frac{1}{2\mathrm{C}_3}(q^2)^2\\ &+ \frac{\mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)}{\mathrm{C}_1}q^1 - \frac{\mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)}{\mathrm{C}_1}q^2 \end{aligned} \tag{4.12}$$

$$D = \mathrm{R}_1 \mathrm{d}q^1. \tag{4.13}$$

The equilibrium point of the considered linear network is the solution of the system

$$\begin{aligned} &\left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_2}\right)q^1 - \frac{1}{\mathrm{C}_1}q^2 + \frac{\mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)}{\mathrm{C}_1} = 0\\ &-\frac{1}{\mathrm{C}_1}q^1 + \left(\frac{1}{\mathrm{C}_1} + \frac{1}{\mathrm{C}_3}\right)q^2 - \frac{\mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)}{\mathrm{C}_1} = 0. \end{aligned} \tag{4.14}$$

If the constants L_1, L_2, C_1, C_2, C_3 satisfy the conditions (3.32), this equilibrium point is **asymptotically stable**. We define a Liapunov function V by

$$\begin{aligned} V(q,\dot{q}) \;=\; &E_{0_\omega}(q,\dot{q}) - E_{0_\omega}(q_e, 0) = \frac{1}{2}\mathrm{L}_1(\dot{q}^1)^2 + \frac{1}{2}\mathrm{L}_2(\dot{q}^2)^2\\ &+ \frac{1}{2\mathrm{C}_1}\left[(q^1 - q^2) - (q_e^1 - q_e^2)\right]^2 + \frac{1}{2\mathrm{C}_2}(q^1 - q_e^1)^2 + \frac{1}{2\mathrm{C}_3}(q^2 - q_e^2)^2 \end{aligned} \tag{4.15}$$

where q_e satisfies the system (4.14). The level curves of the function (4.14) represent a set of ellipsoids surrounding the equilibrium point. Because the Birkhoffian (4.11) is dissipative, it follows that $\mathrm{d}E_{0_\omega} < 0$, and therefore $\mathrm{d}V < 0$.

Let us consider now the case where all the devices are nonlinear; they are described by the relations (3.2). For the coordinate system on $M_{\mathfrak{c}}$ given by (4.5), the Birkhoffian becomes

$$\begin{aligned} \mathfrak{Q}_1(q,\dot{q},\ddot{q}) &= L_1(\dot{q}^1)\ddot{q}^1 + R_1(\dot{q}^1) + C_1(q^1 - q^2 + \mathcal{K}^3) + C_2(q^1)\\ \mathfrak{Q}_2(q,\dot{q},\ddot{q}) &= L_2(\dot{q}^2)\ddot{q}^2 - C_1(q^1 - q^2 + \mathcal{K}^3) + C_3(q^2) \end{aligned} \tag{4.16}$$

with $\mathcal{K}^3 = \mathfrak{c}_1 + \mathfrak{c}_2 + \mathfrak{c}_3 = \mathrm{Q}_1(0) - \mathrm{Q}_2(0) + \mathrm{Q}_3(0)$.

If R_1 satisfies the condition (3.35), the Birkhoffian (4.16) is **dissipative**. The function $E_{0_\omega} : TM_{\mathbf{c}} \to \mathbf{R}$ and the dissipative 1-form satisfying (2.15) are given by

$$\begin{aligned} E_{0_\omega}(q,\dot q) &= \int L_1(\dot q^1)\dot q^1 \mathrm{d}\dot q^1 + \int L_2(\dot q^2)\dot q^2 \mathrm{d}\dot q^2 + \int C_1(q^1 - q^2 + \mathcal{K}^3)(\mathrm{d}q^1 - \mathrm{d}q^2) \\ &\quad + \int C_2(q^1) dq^1 + \int C_3(q^2)\mathrm{d}q^2 - \int\int C_1'(q^1 - q^2 + \mathcal{K}^3)\mathrm{d}q^1 \mathrm{d}q^2 \end{aligned} \tag{4.17}$$

$$D = R_1(\dot q^1)\mathrm{d}q^1. \tag{4.18}$$

The equilibrium points of the considered nonlinear network are the solutions of the system

$$\begin{aligned} &R_1(0) + C_1(q^1 - q^2 + \mathcal{K}^3) + C_2(q^1) = 0 \\ &-C_1(q^1 - q^2 + \mathcal{K}^3) + C_3(q^2) = 0. \end{aligned} \tag{4.19}$$

(1) If $R_1(0) = 0$, and $L_1,\ L_2, C_1,\ C_2,\ C_3$ satisfies (3.42), then the equilibrium points are **locally asymptotically stable**. We define a Liapunov function V by

$$V(q,\dot q) = E_{0_\omega}(q,\dot q) - E_{0_\omega}(q_e, 0) \tag{4.20}$$

with E_{0_ω} given by (4.17) and q_e satisfying (4.19). The Hessian matrix of V at a equilibrium point $(q_e, 0)$ has the expression

$$\mathbf{H}_V(q_e, 0) = \begin{pmatrix} L_1(0) & 0 & 0 & 0 \\ 0 & L_2(0) & 0 & 0 \\ 0 & 0 & \tilde C_1'(q_e^1, q_e^2) + C_2'(q_e^2) & -\tilde C_1'(q_e^1, q_e^2) \\ 0 & 0 & -\tilde C_1'(q_e^1, q_e^2) & \tilde C_1'(q_e^1, q_e^2) + C_3'(q_e^2) \end{pmatrix}. \tag{4.21}$$

Under the assumptions we made, this matrix is positive definite. The centers of the level curves of the function (4.20) have the coordinates $(q_e, 0)$, where q_e satisfies the system (4.19) with $R_1(0) = 0$. Because the Birkhoffian (4.16) is dissipative, it follows that $\mathrm{d}E_{0_\omega} < 0$, and therefore $\mathrm{d}V < 0$.

(2) If $R_1(0) \neq 0$, but for all $x \neq 0$

$$x\,(R_1(x) - R_1(0)) > 0 \tag{4.22}$$

and $L_1,\ L_2, C_1,\ C_2,\ C_3$ satisfies (3.42), then the equilibrium points are **locally asymptotically stable**. We define now a Liapunov function V by

$$\mathrm{V}(q,\dot q) = \mathrm{E}_{0_\omega}(q,\dot q) - \mathrm{E}_{0_\omega}(q_e, 0) \tag{4.23}$$

with $\mathrm{E}_{0_\omega} : TM_{\mathbf{c}} \to \mathbf{R}$ given by

$$\mathrm{E}_{0_\omega}(q,\dot q) = E_{0_\omega}(q,\dot q) + R_1(0)q^1. \tag{4.24}$$

$E_{0_\omega}(q,\dot q)$ has the expression (4.17). The centers of the level curves of the function (4.23) have the coordinates $(q_e, 0)$, where q_e satisfies the system (4.19). The Hessian matrix of the function V in (4.23) has at the equilibrium point the same expression (4.21). It remains to prove that $\mathrm{d}\mathrm{V} < 0$. This is yielded from the dissipativeness of the Birkhoffian (4.16). We consider the following dissipative 1-form:

$$\mathrm{D} = \left[R_1\left(\dot q^1\right) - R_1(0)\right]\mathrm{d}q^1. \tag{4.25}$$

In view of assumption (4.22), this vertical 1-form is indeed dissipative. One can easily check that for the function $\mathrm{E}_{0_\omega}(q,\dot q)$ in (4.24) and the dissipative 1-form in (4.25), the following identity is fulfilled:

$$\sum_{j=1}^{2} Q_j(q\ \dot q,\ \ddot q)\dot q^j = \sum_{j=1}^{2}\left[\frac{\partial \mathrm{E}_{0_\omega}}{\partial q^j}\dot q^j + \frac{\partial \mathrm{E}_{0_\omega}}{\partial \dot q^j}\ddot q^j + \mathrm{D}_j(q,\dot q)\dot q^j\right]. \tag{4.26}$$

The Birkhoffian (4.16) being dissipative we have $\mathrm{d}\mathrm{E}_{0_\omega} < 0$; therefore $\mathrm{d}\mathrm{V} < 0$.

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